

Quantum Metrological Bounds for Vector Parameter in Presence of Noise

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Precise measurement is central to all science and technology. The fundamental bounds on the measurement precision imposed by quantum mechanics are of particular interest in quantum metrology, quantum lithography, gravity-wave detection and quantum computation. The quest to realize increasingly precise measurements requires more knowledge and deeper understanding of these bounds. Here, in view of prior information, we investigate the precision bounds of total mean square error of vector parameter estimation which contains d independent parameters. From quantum Ziv-Zakai error bounds, we derive two kinds of quantum metrological bounds for vector parameter estimation, both of which should be saturated. By these bounds, we show that at best a constant advantage can be obtained via simultaneous estimation strategy over the optimal individual estimation strategy. A general framework for obtaining the lower bounds in the noisy system are also proposed.

Quantum parameter estimation, the emerging field of quantum technology, aims to use entanglement and other quantum resources to yield higher statistical precision of unknown parameters than purely classical approaches[1]. A lot of work has been done, both theoretically[2–5] and experimentally[6–9], to exploit the quantum advantages over classical measurement strategy. Simultaneously estimating more than one parameters represents an interesting possibility to extend the concept of quantum metrology. One application of this new technique of quantum metrology to the wider research community is microscopy. Recently, the quantum enhanced imaging making use of point estimation theory is presented[10], and the vector phase estimation is then investigated since phase imaging is inherently a vector parameter estimation problem[11]. With respect to the mature experimental techniques of multiqubit manipulation and multi-port devices[12], there is an urgent demand for the theoretic study of the multi-mode quantum metrology. The vector parameter estimation technique will also be of significant use for other science and technology areas such as multipartite clock synchronization[13] and gravity-wave detection[14].

Humphreys *et al.*[11] find that quantum simultaneous estimation (SE) strategy provides an advantage in the total variance of a vector parameter over optimal individual estimation (IE) schemes, which remarkably scales as $\mathcal{O}(d)$ with d the number of parameters. This result is obtained via quantum Cramér-Rao bounds (QCRBs) corresponding to the quantum Fisher information (QFI) matrix for unitary evolution in the absence of noise. However, the most well known QCRBs are asymptotically tight in limit of infinitely many trials and can grossly underestimate the achievable error when the likelihood function is highly non-Gaussian[15, 16]. In consideration of the nontrivial prior information of vector parameter, extension of quantum Ziv-Zakai bounds[17] (QZZBs) that relates mean square error to the probability of error in binary detection problem seems a superior alternative in quantum vector parameter estimation.

In this paper, we derive two kinds of quantum metrological bounds for vector parameter with uniform prior distribution by

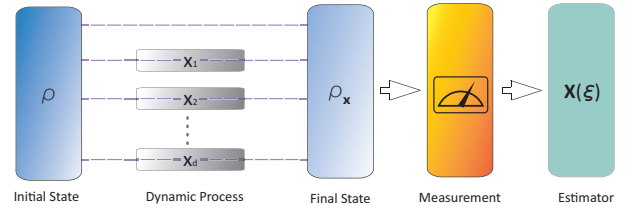


FIG. 1: Vector parameter estimation procedure. d independent dynamic maps corresponding to d parameters \mathbf{x} transform the initial state ρ to the final state $\rho_{\mathbf{x}}$. The vector estimator $\mathbf{X}(\xi)$ is obtained according to results ξ from the measurements performed on the final state $\rho_{\mathbf{x}}$.

extending QZZBs to the vector parameter case. These bounds invoking the quantum speed limit theorem[18] indicate that only a constant advantage can be obtained via SE strategy over the optimal IE strategy with NOON states. In addition to the analysis of optimal probe states, the SE scheme with multimode squeezed vacuum states as a promising optical resource have also been investigated and compared to IE with two-mode squeezed vacuum states. Due to the fact that all realistic experiments face the decoherence, a general framework for obtaining the lower bounds in the noisy system are also proposed. The lower bounds are applied to evaluate the precision under two important decoherence models: photon loss model and phase diffusion model.

Results

Quantum metrological bounds for vector parameter estimation.

In the vector parameter estimation procedure, let $\mathbf{x} = (x_1, \dots, x_m)^T$ be d -dimensional continuous vector random parameter and \mathbf{x} has a priori probability density function (PDF) $P_{\mathbf{x}}(\mathbf{x})$. We may obtain finite measurement results as $\xi = (\xi_1, \xi_2, \dots)^T$ to calculate the vector estimator $\mathbf{X}(\xi) = (X_1(\xi), \dots, X_d(\xi))^T$ with the observation conditional PDF

$P_{\xi|x}(\xi|x)$ of ξ given the true values x . This conditional PDF plays a centre role in the investigation of QFI and QCRB. Instead of the covariance matrix Σ_x discussed in [11], we pay our attention to the estimation error, $\epsilon = X(\xi) - x$, and the mean-square estimation error correlation matrix is defined as [16] $\Sigma_\epsilon = \int dx d\xi P_{x,\xi}(x, \xi) \epsilon \epsilon^T$, where $P_{x,\xi}(x, \xi) = P_{\xi|x}(\xi|x)P_x(x)$ denotes the joint PDF of x and ξ . The total mean square error is obtained by taking a trace of this error correlation matrix

$$|\Sigma_\epsilon| \equiv \text{Tr}(\Sigma_\epsilon) = \sum_{i=1}^d \int dx d\xi P_{x,\xi}(x, \xi) [X_i(\xi) - x_i]^2, \quad (1)$$

with which we will be concerned in the rest of our paper. Ziv-Zakai bounds [15] (ZZBs) assume that the parameter is a random variable with a known priori PDF, while QCRBs treat the parameter as an unknown deterministic vector of quantities. Therefore, QZZBs [17] that relate the mean-square error to the error probability in a binary hypothesis testing problem are superior alternatives for obtaining the lower bounds when taking the prior information into consideration. In this paper, let us assume for the moment that the prior distribution of the vector parameter is a uniform window with mean $\mu = (\mu_1, \dots, \mu_d)^T$ and width $W = (W_1, \dots, W_d)^T$:

$$P_x(x) = \prod_{i=1}^d \frac{1}{W_i} \text{rect}\left(\frac{x_i - \mu_i}{W_i}\right), \quad (2)$$

which means that we have no prior information on the vector parameter before the estimate. This assumption is reasonable because it has been demonstrated that in the high prior information regime, the resulting accuracy is of same order one obtainable by guessing a random value in accordance to the prior PDF [19].

As shown in Fig. 1, general vector parameter estimation procedure can be divided into three distinct sections: probe preparations, interaction between the probe and the system, and the probe readouts for determining estimators [1]. Consider that the interaction between the probe states and the system with the unknown vector parameter can be expressed as a unitary operator $\rho_x = \exp(-iH^T x) \rho \exp(iH^T x)$ where $H = (\hat{H}_1, \dots, \hat{H}_d)^T$ is a vector of Hamiltonians, $[\hat{H}_i, \hat{H}_j] = 0$ and the initial state is a pure state $\rho = |\psi\rangle\langle\psi|$. In this case, the extended QZZBs for vector parameter estimation can be written as the sum of the lower bound for each parameter corresponding to a Hamiltonian:

$$|\Sigma_\epsilon| \geq \sum_{i=1}^d \int_0^{W_i} d\tau_i \frac{\tau_i}{2} \left(1 - \frac{\tau_i}{W_i}\right) \left(1 - \sqrt{1 - F_i^2(\tau_i)}\right), \quad (3)$$

where $F_i(\tau_i) \equiv |\langle\psi|\exp(-i\hat{H}_i\tau_i)|\psi\rangle|$ denotes to the fidelity of a single parameter. Then the problem is translated to evaluation of the fidelity of a single parameter which is a centre and widely studied problem in quantum speed limit theorem [18, 20] and quantum metrology [17]. Thus, it is possible to obtain two kinds of lower bounds using two different approximations [17]: $F_i^2(\tau_i) \geq 1 - 2\lambda\tau_i\langle\hat{H}_i\rangle_+$ and

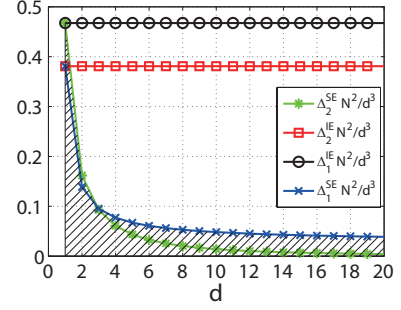


FIG. 2: SE v.s. IE for optimal state. Four bounds, multiplied by N^2/d^3 , of optimal probe state: ML bound Δ_1^{SE} and MT bound Δ_2^{SE} for SE strategy together with ML bound Δ_1^{IE} and MT bound Δ_2^{IE} for IE strategy are compared against d . The hatched area represents the forbidden value of precision for SE strategy.

$F_i^2(\tau_i) \geq \cos^2 \Delta \hat{H}_i \tau_i$, where $\langle\hat{H}_i\rangle_+ \equiv \langle\psi|\hat{H}_i|\psi\rangle - E_i$ with E_i the minimum eigenvalue of \hat{H}_i , $\lambda \simeq 0.7246$ and $\Delta \hat{H}_i^2 \equiv \langle\psi|\hat{H}_i^2|\psi\rangle - \langle\psi|\hat{H}_i|\psi\rangle^2$. Hence, it is possible to obtain a Margolus-Levitin (ML) type bound depending on the effective average energy $|\Sigma_\epsilon| \gtrsim \sum_{i=1}^d \frac{1}{80\lambda^2 \langle\hat{H}_i\rangle_+^2}$ for $W_i \gg 1/(2\lambda\langle\hat{H}_i\rangle_+)$ and a Mandelstam-Tamm (MT) type bound limited by the fluctuation of the Hamiltonian: $|\Sigma_\epsilon| \gtrsim \sum_{i=1}^d \frac{\pi^2/16 - 1/2}{\Delta \hat{H}_i^2}$ for $W_i \gg \pi/(2\Delta \hat{H}_i)$. The MT type bound is capable of predicting the same scaling with QCRBs of multiple parameter estimation [11], however, may be less tight than QCRBs for some cases. The ML type bound scales with the average energy relative to the ground state is a new bound for multiple parameter estimation. Moreover, we will show that the ML type bound exists even if the ground energy of Hamiltonian is negative infinite or energy eigenvalues are continuous. It is necessary to emphasize that both bounds, ML type or MT type, include the prior information and should be saturated simultaneously. Therefore, when $W_i \gg \max\{1/(2\lambda\langle\hat{H}_i\rangle_+), \pi/(2\Delta \hat{H}_i)\}$ holds, the acceptable lower bound from QZZBs is written as

$$|\Sigma_\epsilon| \gtrsim \max \left\{ \sum_{i=1}^d \frac{c_{ML}}{\langle\hat{H}_i\rangle_+^2}, \sum_{i=1}^d \frac{c_{MT}}{\Delta \hat{H}_i^2} \right\}, \quad (4)$$

where we let $c_{ML} = 1/80\lambda^2$ and $c_{MT} = \pi^2/16 - 1/2$. Letting $d = 1$, these lower bounds will reduce into the results discussed in [17] which conform to the mainstream understanding of the Heisenberg limit [5, 21].

From equation (4), the vector parameter estimation problem is decomposed into d single parameter estimation problems and the solution of each problem is affordable by the investigation on the effective average and the variance of the Hamiltonian of each mode given the same state. This result is of great importance since the conclusions from the study of single parameter estimation give direct operational significance to the vector parameter estimation.

Optimal probe states and multimode squeezed states without noises.

Recently, it is derived from QCRBs that the advantage of SE over the optimal IE is at best $\mathcal{O}(d)$ [11]. Here we investigate the performance of SE against IE when taking the prior information of vector parameter into consideration.

Hamiltonian operators are considered as $H_i = \hat{n}_i$ and we label the optimal probe state of totally N particles in the Fock space as [11]

$$|\psi_o\rangle = \beta|N, \underbrace{0, \dots, 0}_d\rangle + \alpha|0, N, 0, \dots, 0\rangle + \dots + |0, \dots, 0, N\rangle \quad (5)$$

with $d\alpha^2 + \beta^2 = 1$ for $\alpha = 1/\sqrt{d + \sqrt{d}}$. The effective average and the variance for the i th Hermitian can be calculated as $\langle \hat{n}_i \rangle_+ = \alpha^2 N$ and $\Delta \hat{n}_i^2 = \alpha^2(1 - \alpha^2)N^2$ which lead to the lower bounds for SE strategy as

$$|\Sigma_\epsilon| \geq \max \left\{ \frac{d(d + \sqrt{d})^2 c_{ML}}{N^2}, \frac{d(d + \sqrt{d})^2 c_{MT}}{(d + \sqrt{d} - 1)N^2} \right\}. \quad (6)$$

Here the first bound is a ML type bound labelled as Δ_1^{SE} and the second one is a MT type lower bound labelled as Δ_2^{SE} . The best quantum strategy of IE uses NOON states with total N/d photons for each parameter. Thus, the ML bound for IE is $\Delta_1^{IE} \simeq d^3/(20\lambda^2 N^2)$ and the MT for IE is $\Delta_2^{IE} \simeq d^3(\pi^2 - 8)/(4N^2)$. The combined bounds for SE strategy and IE strategy uniformly multiplied by N^2/d^3 are compared in Fig. 2. For IE strategy, it is easy to conclude that MT type bound Δ_2^{IE} is tighter. For SE strategy, the hatched area represents the forbidden value due to the combined bound (6)-for large value of d , ML bound is tighter, while MT bound is tighter for small value of d . Although, for large d , the MT bound for SE presents a $\mathcal{O}(d)$ advantage over both types of bounds for IE strategy, the ML bound that should also be fulfilled denies this advantage and merely a constant advantage ($\lim_{d \rightarrow \infty} \Delta_2^{IE}/\Delta_1^{SE} \simeq 4.9081$) can be observed. Generally, it is possible to demonstrate that for any multiparticle entangled state with N total particle number, no $\mathcal{O}(d)$ advantage can be gained via SE strategy over IE strategy.

Another entangled state we focus on is $(d + 1)$ -mode squeezed vacuum states[22, 23] which may not provide a more accurate measurement of vector parameter but is more promising in the real application with existing experimental techniques. Recently, it has been reported that advanced LIGO can be improved with squeezed vacuum states [14]. Whether $(d + 1)$ -mode squeezed vacuum state can provide advantage over the IE strategy via using two-mode squeezed vacuum state to estimate each parameter individually is an interesting and important problem.

Using the relation between the Bose operators $(\hat{a}_i, \hat{a}_i^\dagger)$, the coordinate operator $\hat{Q}_i = (\hat{a}_i + \hat{a}_i^\dagger)/\sqrt{2}$, and the momentum operator $\hat{P}_i = (\hat{a}_i - \hat{a}_i^\dagger)/(i\sqrt{2})$, one can express the $(d + 1)$ -mode squeezed operator $\hat{S}(r)$ with form[23]

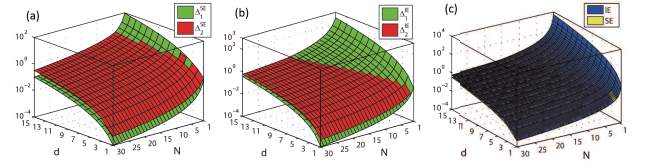


FIG. 3: SE v.s. IE for multimode squeezed vacuum state. (a) ML bound Δ_1^{SE} and MT bound Δ_2^{SE} for SE strategy using the $(d + 1)$ -mode squeezed vacuum state against d and total average particle number N . (b) ML bound Δ_1^{IE} and MT bound Δ_2^{IE} for IE strategy using d two-mode squeezed vacuum states against d and total average particle number N . (c) The combined bounds for SE strategy and the one for IE strategy are compared against d and total average particle number N .

$\hat{S}(r) = \exp(ir\mathbf{Q}^T \mathbf{A} \mathbf{P})$ where $\mathbf{Q} = (\hat{Q}_0, \hat{Q}_1, \dots, \hat{Q}_d)^T$, $\mathbf{P} = (\hat{P}_0, \hat{P}_1, \dots, \hat{P}_d)^T$ and \mathbf{A} is a $(d + 1) \times (d + 1)$ matrix with elements $A_{kj} = \delta_{k, (j+1) \bmod (d+1)}$ and $k, j = 0, 1, \dots, d$. Our SE strategy using the squeezed vacuum state $\hat{S}(r)|\underbrace{0, \dots, 0}_{d+1}\rangle$ as the probe state is compared with the IE

strategy using two-mode squeezed vacuum state. The total average particle numbers for the $(d + 1)$ -mode squeezed vacuum state and d identical two-mode squeezed vacuum states are set the same. Numerical results of lower bounds for SE and IE are presented in Fig. 3(a),(b) and are compared in Fig. 3(c). These results indicate that the SE strategy with $d + 1$ -mode squeezed vacuum state is superior to the IE strategy except for the circumstances where $d = 2$ and the value of total average photon number N is around 2.

Quantum metrological bounds in noisy systems.

The quantum metrological bounds in noisy systems have become a focus of attention because in real experiments there will always be some degree of noise and limitation. For unitary processes, the analytical expressions of the lower bound for estimating multiple parameter have been established, however, for noisy case, the task may be of exceptional difficulty since saturation of QFI matrix for vector parameters is, in general, difficult or impossible.

Here we generalize our analysis to vector parameter estimation in noisy system. Since the noisy dynamic process is assumed to include d independent quantum channels corresponding to d parameters, respectively, it can be described as $\rho_{\mathbf{x}} = \sum_{\mathbf{l}} \hat{\Pi}_{\mathbf{l}}(\mathbf{x}) \rho \hat{\Pi}_{\mathbf{l}}^\dagger(\mathbf{x})$ in terms of Kraus operators $\{\hat{\Pi}_{\mathbf{l}}(\mathbf{x}) = \bigotimes_{i=1}^d \hat{\pi}_{l_i}^{(i)}(x_i)\}$ with $\mathbf{l} = (l_1, l_2, \dots, l_d)$ under identity condition $\sum_{\mathbf{l}} \hat{\pi}_{l_i}^{(i)\dagger}(x_i) \hat{\pi}_{l_i}^{(i)}(x_i) = \mathbb{I}$. This dynamic process has another equivalent description expressed as tracing environment after a unitary evolution operators $U_{SE}(\mathbf{x}) = \bigotimes_{i=1}^d U_{SE}^{(i)}(x_i)$ acting on the pure state $|\psi\rangle|0\rangle_E$ of an enlarged space for the system interacting with an environment. With these preconditions, when the Hermitian operator

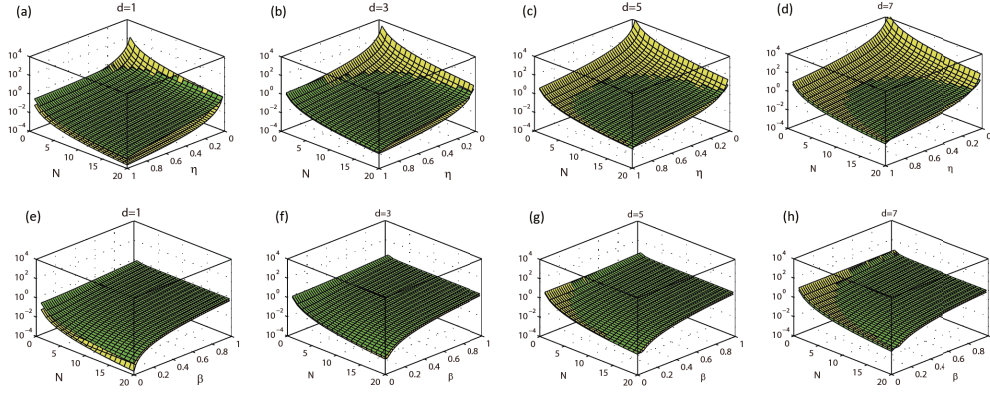


FIG. 4: Noisy ML bound v.s. MT bound for optimal probe state. Green curved surface is for Δ_2^{SE} and the yellow one is for Δ_1^{SE} . (a)-(d) are for the photon loss model: two bounds for state (5) are compared against particle number N and intensity of the transmissivity η . (e)-(h) are for the phase diffusion model: two bounds for state (5) are compared against particle number N and diffusion parameter η .

for the i th mode [25]:

$$\hat{\mathcal{H}}_i(x_i) \equiv -i \frac{d[u_E^{(i)}(x_i)U_{SE}^{(i)}(x_i)]^\dagger}{dx_i} u_E^{(i)}(x_i) U_{SE}^{(i)}(x_i) \quad (7)$$

is independent of the parameter x_i , a rational twifomed lower bound for noisy multiple parameter estimation can be simply written as

$$|\Sigma_\epsilon| \geq \max \left\{ \sum_{i=1}^d \frac{c_{ML}}{\min \langle \hat{\mathcal{H}}_i \rangle_+^2}, \sum_{i=1}^d \frac{c_{MT}}{\min \Delta \hat{\mathcal{H}}_i^2} \right\} \quad (8)$$

where the average is taken on state $|\psi\rangle|0\rangle_E$, and the minimum runs over all the possible forms of the unitary operator $u_E^{(i)}(x_i)$ acting on the environment of i th mode. Here, the unitary matrix $\bigotimes_{i=1}^d \hat{u}_E^{(i)}(x_i)$ relates different purifications of the final states ρ_x and connects different sets of linearly independent Kraus operators[26], as well. For MT bound, the minimum variance of $\hat{\mathcal{H}}_i$ multiplied by 4 equals to the attainable QFI of i th parameter alone in noisy system and has been discussed in[24, 25]. ML bound in presence of the noise that corresponds to the effective average of $\hat{\mathcal{H}}_i$ is a new result even for the single parameter estimation. It concerns with the residual resource for estimating the parameter after suffering the noise. In addition, we should point out that the optimal unitary matrix that saturates the minimum of ML bound need not be the same as the optimal one for MT bound.

To demonstrate the practicability of these lower bounds, we consider two common but significant models in the noisy optical interferometry: photon loss model and phase diffusion model. The derivations of the lower bounds for these two models are presented in Method. For simplicity, we suppose that different modes of lossy channels share the same lossy parameter: for the photon loss model, the intensity of the transmissivity of i th mode has $\eta_i = \eta$, and for phase diffusion model, the diffusion parameter of i th mode is set as $\beta_i = \beta$. For different values of d , the ML bounds (yellow curved surface) and MT bounds (green curved surface), given

state (5), are compared in Fig. 4. (Fig. 4(a)-(d) are for photon loss model and Fig. 4(e)-(h) are for phase diffusion model.) It is clearly displayed that under some conditions MT bound are tighter than ML bound and vice versa. It shows that both bounds MT type or ML type are essential and need to be saturated when taking these noise models into account. In particular, our lower bounds that include the prior information of vector parameter are shown to be versatile in different quantum metrology problems and can be much tighter than the popular QCRB in certain cases.

Discussions

In this paper, we extend QZZBs to the multiple parameter case and present two kinds of lower bounds in accordance with the quantum limit theorem. Compared with QCRBs depending on the infinitesimal statistical distance between ρ_x and its neighborhood[2], QZZBs depend on the statistical distance between ρ_x and $\rho_{x'}$ for all relevant values of x and x' . That is, our method provides a lower bound on the achievable precision in consideration of the PDF characterizing the prior information of the vector parameter, which considers a more realistic estimation problem.

An important question not addressed above is the attainability of our twifomed lower bound. This saturation problem appears tough in two aspects. One is that, different from QCRBs in special cases, the QZZBs are not excepted to be saturated and can not be used to study the optimal performance of quantum parameter estimation[15]. However, the QZZBs are shown to be much more versatile and tighter than the popular QCRBs in many cases[17] such as single parameter estimation with squeezed states and vector parameter estimation. Another aspect is that, the twifomed bound borrowing ideas from the quantum limit theorem uses some approximations that may not be saturated. Nevertheless, our bounds limited by both average and variance of the Hamiltonian present a much clearer physical meaning which also includes that of QCRBs.

A tighter bound can be obtained by the analytical or numerical studies on the extended QZZBs in equation (3).

Assessing the impact of noise on the performance of SE strategy for vector parameter estimation is a crucial problem in quantum metrology and quantum imaging. It seems indeed difficult as the attainability of QCRBs for noisy vector parameter estimation is not resolved or may be impossible. Our investigation on the lower bounds in noisy system will perform a new tool for evaluating noisy quantum metrology, even though it is not tight either. Moreover, unlike QCRBs, the results from single parameter estimation will contribute directly to the case of vector parameter in noisy systems.

Methods

Quantum Ziv-Zakai bounds for vector parameter estimation.

We extend QZZBs to solving the vector parameter problem and one version of the extended QZZBs are derived with the valley-filling operation $\mathcal{V}f(\tau_i) \equiv \max_{\eta \geq 0} f(\tau_i + \eta)$ as

$$|\Sigma_\epsilon| \geq \sum_{i=1}^d \mathcal{V} \int_0^\infty d\tau_i \frac{\tau_i}{2} \int d\mathbf{x} \min\{P_{\mathbf{x}}(\mathbf{x}), P_{\mathbf{x}}(\mathbf{x} + \boldsymbol{\tau}_i)\} \times \left(1 - \sqrt{1 - F^2(\rho_{\mathbf{x}}, \rho_{\mathbf{x} + \boldsymbol{\tau}_i})}\right) \quad (9)$$

where $F(\rho, \sigma) = \text{tr}(\sqrt{\rho^{1/2}\sigma\rho^{1/2}})$ refers to the Brues' fidelity between two density matrices ρ and σ , and we should define $\boldsymbol{\tau}_i = (\underbrace{0, \dots, 0}_{i-1}, \tau_i, 0, \dots, 0)^T$. Another slightly tighter

version[19] of QZZB for multiple parameters case is written as

$$|\Sigma_\epsilon| \geq \sum_{i=1}^d \mathcal{V} \int_0^\infty d\tau_i \frac{\tau_i}{2} \int d\mathbf{x} [P_{\mathbf{x}}(\mathbf{x}) + P_{\mathbf{x}}(\mathbf{x} + \boldsymbol{\tau}_i)] \times \left(1 - \sqrt{1 - 4P_0P_1F^2(\rho_{\mathbf{x}}, \rho_{\mathbf{x} + \boldsymbol{\tau}_i})}\right) \quad (10)$$

where $P_0 = P_{\mathbf{x}}(\mathbf{x})/[P_{\mathbf{x}}(\mathbf{x}) + P_{\mathbf{x}}(\mathbf{x} + \boldsymbol{\tau}_i)]$ and $P_1 = 1 - P_0$ are probabilities of states $\rho_{\mathbf{x}}$ and $\rho_{\mathbf{x} + \boldsymbol{\tau}_i}$. These two bounds are equivalent when considering that the vector parameter has a uniform prior distribution as presented in equation (2).

Mean and variance of number operator in multi-mode squeezed vacuum state.

Using the Campbell-Baker-Hausdorff formula we see that the squeezing operator transforms the annihilation operators as

$$\hat{S}^\dagger(r)\hat{a}_k\hat{S}(r) = \sum_{i=0}^d \left(R_{ki}\hat{a}_i + T_{ki}\hat{a}_i^\dagger\right) \quad (11)$$

where $R = (e^{-r\tilde{A}} + e^{rA})/2$, $T = (e^{-r\tilde{A}} - e^{rA})/2$ and $\tilde{A} \equiv A^T$. Because we have that $\tilde{A}^D = A^D = \mathbb{I}$ with $D = d + 1$, it

can be expanded as $\exp(-r\tilde{A}) = c_0(r)\mathbb{I} + c_1(r)\tilde{A} + c_2(r)\tilde{A}^2 + \dots + c_d(r)\tilde{A}^d$, and the coefficients are given by the following equation:

$$\begin{pmatrix} c_0(r) \\ c_1(r) \\ \vdots \\ c_d(r) \end{pmatrix} = \frac{1}{D} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & e^{-i\omega_D} & \cdots & e^{-id\omega_D} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & e^{-id\omega_D} & \cdots & e^{-id^2\omega_D} \end{pmatrix} \cdot \begin{pmatrix} \exp(-r) \\ \exp(-re^{i\omega_D}) \\ \vdots \\ \exp(-re^{id\omega_D}) \end{pmatrix}, \quad (12)$$

where $\omega_D = 2\pi/D$. These give for the following expectation and variance of number operator of the k th mode:

$$\langle \hat{n}_k \rangle = \frac{1}{4} \sum_{i=0}^d [c_i^2(r) + c_i^2(-r)] - \frac{1}{2}, \quad (13)$$

and

$$\Delta \hat{n}_k^2 = \frac{1}{8} \left(\sum_{i=0}^d c_i^2(r) \right)^2 + \frac{1}{8} \left(\sum_{i=0}^d c_i^2(-r) \right)^2 - \frac{1}{4}, \quad (14)$$

with which we numerically evaluate the performance of multi-mode squeezed state in the multiple parameter estimation procedure.

Photon loss model.

Due to Uhlmann's theorem, the fidelity is bounded by

$$F^2(\rho_{\mathbf{x}}, \rho_{\mathbf{x} + \boldsymbol{\tau}_i}) \geq \left| {}_E \langle \mathbf{0} | \langle \psi | U_{SE}^\dagger(\mathbf{x}) U_{SE}(\mathbf{x} + \boldsymbol{\tau}_i) | \psi \rangle | \mathbf{0} \rangle_E \right|^2. \quad (15)$$

Assume for the moment that Hermitian operator expressed in (7) is independent of the parameter x_i , the approximations applied in the unitary case will still hold and the bounds (8) can be easily obtained, where the minimums make the bound tight for all the equivalent descriptions of the dynamic process.

A possible set of Kraus operators describing the photon loss model in i th mode is written as[24]:

$$\hat{\kappa}_{l_i}^{(i)} = \sqrt{\frac{(1-\eta)^{l_i}}{l_i!}} e^{ix_i(\hat{n}_i - \delta_i l_i)} \eta_i^{\frac{\hat{n}_i}{2}} \hat{a}_i^{l_i}, \quad i = 1, \dots, d, \quad (16)$$

given $\hat{n}_i = \hat{a}_i^\dagger \hat{a}_i$ the number operator on i th mode, η_i the intensity of the transmissivity and δ_i the variational parameter. The effective average of the Hermitian operator (7) is $\langle \hat{\mathcal{H}}_i^{pl} \rangle_+ = \langle \hat{n}_i \rangle [1 - \sigma_i(1 - \eta_i)] - \min\{N(1 - \sigma_i), 0\}$ and the variance is $\Delta(\hat{\mathcal{H}}_i^{pl})^2 = \Delta \hat{n}_i^2 [1 - \sigma_i(1 - \eta_i)]^2 + \langle \hat{n}_i \rangle \sigma_i^2 \eta_i (1 - \eta_i)$, where we let $\sigma_i = \delta_i + 1$, and $\min\{N(1 - \sigma_i), 0\}$ denotes the ground energy eigenvalue. The minimums of these two

quantities for the optimal variational parameter contribute to the lower bound for photon loss channel:

$$|\Sigma_\epsilon|^{pl} \geq \max \left\{ \sum_{i=1}^d \frac{c_{ML}}{\eta_i^2 \langle \hat{n}_i \rangle^2}, \sum_{i=1}^d \frac{c_{MT}}{(1-\eta_i) \Delta \hat{n}_i^2 + \eta_i \langle \hat{n}_i \rangle} \right\} \quad (17)$$

where $\delta_i^{opt} = 1$ is the optimal variational parameter for ML type bound, and $\delta_i^{opt'} = \frac{\Delta \hat{n}_i^2}{(1-\eta_i) \Delta \hat{n}_i^2 + \eta_i \langle \hat{n}_i \rangle} - 1$ is for MT type bound.

Phase diffusion model.

Under the situation that $\sqrt{2}\beta^2 n \gg 1$ [25], the phase diffusion noise for d independent modes can be represented, in the Markov limit, by the unitary for system and environment as

$$u_E(\mathbf{x}) U_{SE}(\mathbf{x}) = \bigotimes_{i=1}^d e^{-ix_i \hat{n}_i} e^{ix_i \kappa_i \hat{P}_i^E / 2\beta_i} e^{i2\beta_i \hat{n}_i \hat{Q}_i^E} \quad (18)$$

where $\hat{Q}_i^E = (\hat{b}_i + \hat{b}_i^\dagger)/\sqrt{2}$ and $\hat{P}_i = (\hat{b}_i - \hat{b}_i^\dagger)/(i\sqrt{2})$ with \hat{b}_i and \hat{b}_i^\dagger the annihilation and creation operators acting on the environment corresponding to the i th mode of the system. For the i th mode, β_i stands for the diffusion parameter and κ_i is the variational parameter. In this model, the Hermitian operator for the enlarged system can be explicitly expressed as

$$\hat{\mathcal{H}}_i^{pd} = (1 - \kappa_i) \hat{n}_i - \kappa_i \hat{P}_i^E / (2\beta_i). \quad (19)$$

We can follow [25] and obtain the minimum variance of the i th mode as $\Delta(\hat{\mathcal{H}}_i^{pd})^2 = \frac{\Delta \hat{n}_i^2}{1+8\beta_i^2 \Delta \hat{n}_i^2}$ for optimal variational parameter $\kappa_i = 8\Delta \hat{n}_i^2 \beta_i^2 / (1 + 8\Delta \hat{n}_i^2 \beta_i^2)$. However, for the ML bound, the calculation of the effective average may be impossible because the eigenvalues of $\hat{\mathcal{H}}_i^{pd}$ are continuous and the ground energy eigenvalue is negative infinity. In order to obtain the appropriate and nontrivial form of $\langle \hat{\mathcal{H}} \rangle_+$, we restart from the form of the fidelity and obtain another version of the effective average which may lead to a less tight bound, but contains a clear physical implication. The effective average for this model is calculated as (For details, please see Supplementary Information) $\langle \hat{\mathcal{H}}_i^{pd} \rangle'_+ = [1 - \kappa_i] \langle \hat{n}_i \rangle + |\kappa_i| / (2\sqrt{2}\pi\beta_i)$ and the ML bound that contains a optimal problem is studied by numerical simulations and shown in Fig. 4(e)-(h).

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- [1] Giovannetti, V., Lloyd, S. & Maccone, L. Advances in quantum metrology. *Nature Photon.* **5**, 222 (2011).
- [2] Braunstein, S. L. & Caves, C. M. Statistical distance and the geometry of quantum states. *Phys. Rev. Lett.* **72**, 3439 (1994).
- [3] Giovannetti, V., Lloyd, S. & Maccone, L. Quantum-enhanced positioning and clock synchronization. *Nature* **412**, 417 (2001).
- [4] Lee, H., Kok, P. & Dowling, J. P. A quantum Rosetta stone for interferometry. *J. Mod. Opt.* **49**, 2325 (2002).
- [5] Hyllus, P., Pezzé, L. & Smerzi, A. Entanglement and sensitivity in precision measurements with states of a fluctuating number of particles. *Phys. Rev. Lett.* **105**, 120501 (2010).
- [6] Mitchell, M. W., Lundeen, J. S. & Steinberg, A. M. Super-resolving phase measurements with a multiphoton entangled state. *Nature* **429**, 161 (2004).
- [7] Leibfried, D. *et al.* Toward Heisenberg-limited spectroscopy with multiparticle entangled states. *Science* **304**, 1476 (2004).
- [8] Nagata, T. *et al.* Beating the standard quantum limit with four-entangled photons. *Science* **316**, 726 (2007).
- [9] Xiang, G. Y. *et al.* Entanglement-enhanced measurement of a completely unknown optical phase. *Nature Photon.* **5**, 43 (2011).
- [10] Petéz-Delgado, C. A., Pearce, M. E. & Kok, P. Fundamental limits of classical and quantum imaging. *Phys. Rev. Lett.* **109**, 123601 (2012).
- [11] Humphreys, P. C. *et al.* Quantum enhanced multiple phase estimation. *Phys. Rev. Lett.* **111**, 070403 (2013).
- [12] Peruzzo, A. *et al.* Multimode quantum interference of photons in multiport integrated devices. *Nature Commun.* **2**, 244 (2011).
- [13] Jozsa, R. *et al.* Quantum clock synchronization based on shared prior entanglement. *Phys. Rev. Lett.* **85**, 2010 (2000).
- [14] Aasi, J. *et al.* Enhanced sensitivity of the LIGO gravitational wave detector by using squeezed states of light. *Nature Photon.* **7**, 613 (2013).
- [15] Van Trees, H. L. & Bell, K. L. *Bayesian bounds for parameter estimation and nonlinear filtering/tracking* (Wiley-IEEE, Piscataway, New York, 2007).
- [16] Bell, K. L. *et al.* Extended Ziv-Zakai lower bound for vector parameter estimation. *IEEE Transactions on information theory* **43**, 624 (1997).
- [17] Tsang, M. Ziv-Zakai error bounds for quantum parameter estimation. *Phys. Rev. Lett.* **108**, 230401 (2012).
- [18] Giovannetti, V., Lloyd, S. & Maccone, L. Quantum limits to dynamical evolution. *Phys. Rev. A* **67**, 052109 (2003).
- [19] Giovannetti, V. & Maccone, L. Sub-Heisenberg estimation strategies are ineffective. *Phys. Rev. Lett.* **108**, 210404 (2012).
- [20] Deffner, S. & Lutz, E. Quantum speed limit for non-Markovian dynamics. *Phys. Rev. Lett.* **111**, 010402 (2013).
- [21] Giovannetti, V., Lloyd, S. & Maccone, L. Quantum measurement bounds beyond the uncertainty relations. *Phys. Rev. Lett.* **108**, 260405 (2012).
- [22] Fan, H. Y. & Yu, G. C. Three-mode squeezed vacuum state in fock space as an entangled state. *Phys. Rev. A* **65**, 033829 (2002).
- [23] Ren, G. *et al.* An N-mode squeezed vacuum state in Fock space as an entangled state. *J. Phys. A: Math. Theor.* **40**, 10311 (2007).
- [24] Escher, B. M. *et al.* General framework for estimating the ultimate precision limit in noisy quantum-enhanced metrology. *Nature Phys.* **7**, 406 (2011).
- [25] Escher, B. M. *et al.* Quantum metrological limits via a variational approach. *Phys. Rev. Lett.* **109**, 190404 (2012).
- [26] Demkowicz-Dobrzański, R. *et al.* The elusive Heisenberg limit in quantum enhanced metrology. *Nature Commun.* **3**, 1063 (2012).

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SUPPLEMENTARY INFORMATION: QUANTUM METROLOGICAL BOUNDS FOR VECTOR PARAMETER ESTIMATION IN PRESENCE OF NOISE

I. QUANTUM ZIV-ZAKAI BOUNDS FOR VECTOR PARAMETER ESTIMATION

Let $\epsilon_i \equiv |X_i(\boldsymbol{\xi}) - x_i|$ be a nonnegative random variable. The probability density of ϵ_i is the differentiation of cumulative probability:

$$P_{\epsilon_i}(s_i) = \frac{d}{ds_i} \Pr(\epsilon_i < s_i) = -\frac{d}{ds_i} \Pr(\epsilon_i \geq s_i). \quad (\text{S1})$$

Therefore, one obtains that

$$|\boldsymbol{\Sigma}_\epsilon| = \sum_{i=1}^d \int_0^\infty ds_i s_i^2 P_{\epsilon_i}(s_i) = \sum_{i=1}^d \int_0^\infty d\tau_i \frac{\tau_i}{2} \Pr\left(|X_i(\boldsymbol{\xi}) - x_i| \geq \frac{\tau_i}{2}\right) \quad (\text{S2})$$

where $\tau_i = 2s_i$. Here, one can easily obtain that

$$\Pr\left(|X_i(\boldsymbol{\xi}) - x_i| \geq \frac{\tau_i}{2}\right) = \Pr\left(X_i(\boldsymbol{\xi}) \geq x_i + \frac{\tau_i}{2}\right) + \Pr\left(X_i(\boldsymbol{\xi}) \leq x_i - \frac{\tau_i}{2}\right) \quad (\text{S3})$$

$$= \int d\boldsymbol{\zeta} [P_{\mathbf{x}}(\boldsymbol{\zeta}) + P_{\mathbf{x}}(\boldsymbol{\zeta} + \boldsymbol{\tau}_i)] \left[P_i^0 \Pr\left(X_i(\boldsymbol{\xi}) \geq \zeta_i + \frac{\tau_i}{2} \middle| \mathbf{x} = \boldsymbol{\zeta}\right) + P_i^1 \Pr\left(X_i(\boldsymbol{\xi}) \leq \zeta_i + \frac{\tau_i}{2} \middle| \mathbf{x} = \boldsymbol{\zeta} + \boldsymbol{\tau}_i\right) \right] \quad (\text{S4})$$

where $P_i^0 = \frac{P_{\mathbf{x}}(\boldsymbol{\zeta})}{P_{\mathbf{x}}(\boldsymbol{\zeta}) + P_{\mathbf{x}}(\boldsymbol{\zeta} + \boldsymbol{\tau}_i)}$, $P_i^1 = \frac{P_{\mathbf{x}}(\boldsymbol{\zeta} + \boldsymbol{\tau}_i)}{P_{\mathbf{x}}(\boldsymbol{\zeta}) + P_{\mathbf{x}}(\boldsymbol{\zeta} + \boldsymbol{\tau}_i)}$, $\boldsymbol{\zeta} = (\zeta_1, \dots, \zeta_d)^T$ and $\boldsymbol{\tau}_i = (\underbrace{0, \dots, 0}_{i-1}, \tau_i, 0, \dots, 0)^T$.

Now let us consider a binary hypothesis testing problem with two hypotheses (see Table I). The i th error probability (two kinds of error decision) of the binary hypothesis testing problem can be written as [17]

$$\Pr_e(\zeta_i, \zeta_i + \tau_i) = P_i^0 \Pr\left(X_i(\boldsymbol{\xi}) \geq \zeta_i + \frac{\tau_i}{2} \middle| \mathbf{x} = \boldsymbol{\zeta}\right) + P_i^1 \Pr\left(X_i(\boldsymbol{\xi}) \leq \zeta_i + \frac{\tau_i}{2} \middle| \mathbf{x} = \boldsymbol{\zeta} + \boldsymbol{\tau}_i\right) \quad (\text{S5})$$

and is bounded by the minimum error probability of the hypothesis testing problem, denoted by $\mathcal{P}_e(\zeta_i, \zeta_i + \tau_i)$ which does not depend on $X_i(\boldsymbol{\xi})$. Thus, one obtains that

$$\Pr\left(|X_i(\boldsymbol{\xi}) - x_i| \geq \frac{\tau_i}{2}\right) \geq \int d\boldsymbol{\zeta} [P_{\mathbf{x}}(\boldsymbol{\zeta}) + P_{\mathbf{x}}(\boldsymbol{\zeta} + \boldsymbol{\tau}_i)] \mathcal{P}_e(\zeta_i, \zeta_i + \tau_i). \quad (\text{S6})$$

As $\Pr\left(|X_i(\boldsymbol{\xi}) - x_i| \geq \frac{\tau_i}{2}\right)$ is a monotonically decreasing function of τ_i , a tighter bound can be obtained if we fill the valleys of the right-hand side as a function of τ_i . Denoting this valley-filling operation as $\mathcal{V}f(\tau_i) \equiv \max_{\eta \geq 0} f(\tau_i + \eta)$, with which one obtains the Ziv-Zakai bounds for vector parameter estimation:

$$|\boldsymbol{\Sigma}_\epsilon| \geq \sum_{i=1}^d \int_0^\infty d\tau_i \frac{\tau_i}{2} \mathcal{V} \int d\boldsymbol{\zeta} [P_{\mathbf{x}}(\boldsymbol{\zeta}) + P_{\mathbf{x}}(\boldsymbol{\zeta} + \boldsymbol{\tau}_i)] \mathcal{P}_e(\zeta_i, \zeta_i + \tau_i) \quad (\text{S7})$$

Another version that relates the mean-square error to an equally-likely-hypothesis-testing problem: $P_i^0 = P_i^1 = 1/2$. The Ziv-Zakai bound follows

$$|\boldsymbol{\Sigma}_\epsilon| \geq \sum_{i=1}^d \int_0^\infty d\tau_i \tau_i \mathcal{V} \int d\boldsymbol{\zeta} \min\{P_{\mathbf{x}}(\boldsymbol{\zeta}), P_{\mathbf{x}}(\boldsymbol{\zeta} + \boldsymbol{\tau}_i)\} \mathcal{P}_e^{el}(\zeta_i, \zeta_i + \tau_i) \quad (\text{S8})$$

TABLE I: Binary hypothesis testing problem

Two hypotheses	$\mathcal{H}_i^0 : \mathbf{x} = \boldsymbol{\zeta}$	$\mathcal{H}_i^1 : \mathbf{x} = \boldsymbol{\zeta} + \boldsymbol{\tau}_i$
Probability	$\Pr(\mathcal{H}_i^0) = P_i^0$	$\Pr(\mathcal{H}_i^1) = P_i^1$
Estimation standard ^a	$\delta_i^0 : X_i(\boldsymbol{\xi}) \leq \zeta_i + \frac{\tau_i}{2}$	$\delta_i^1 : X_i(\boldsymbol{\xi}) \geq \zeta_i + \frac{\tau_i}{2}$

^aThe i th estimation standard (totally d standards).

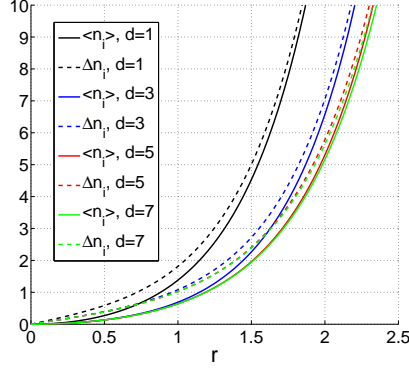


FIG. 5: $\langle n_i \rangle$ and Δn_i against r for multimode squeezed vacuum with $d = 1, 3, 5, 7$.

where $\mathcal{P}_e^{el}(\zeta_i, \zeta_i + \tau_i)$ denotes the minimum error probability for equally likely hypotheses. If $P_{\mathbf{x}}(\zeta)$ is a uniform window, two bounds are equivalent. From Eq. (S5), we define d pairs of projectors: $\Pi_i^0 + \Pi_i^1 = \mathbb{I}$ which denote that

$$\Pr \left(X_i(\xi) \geq \zeta_i + \frac{\tau_i}{2} \middle| \mathbf{x} = \zeta \right) = \text{Tr}(\rho_{\zeta} \Pi_i^1) \quad (\text{S9})$$

$$\Pr \left(X_i(\xi) \leq \zeta_i + \frac{\tau_i}{2} \middle| \mathbf{x} = \zeta + \tau_i \right) = \text{Tr}(\rho_{\zeta + \tau_i} \Pi_i^0). \quad (\text{S10})$$

Then, Eq. (S5) can be rewritten as $\Pr_e(\zeta_i, \zeta_i + \tau_i) = P_i^0 + \text{Tr}(\Gamma_i \Pi_i^0)$, where $\Gamma_i = P_i^1 \rho_{\zeta + \tau_i} - P_i^0 \rho_{\zeta}$. We can write Γ_i in terms of its eigenvalues, which may be positive and negative: $\Gamma_i = \sum_j \gamma_i^j |j\rangle_i \langle j|$, and the minimum error probability is obtained by choosing $\Pi_i^0 = \sum_{j: \gamma_i^j \leq 0} |j\rangle_i \langle j|$:

$$\mathcal{P}_e(\zeta_i, \zeta_i + \tau_i) = P_i^0 - \frac{\|\Gamma_i\| - \text{Tr}(\Gamma_i)}{2} = \frac{1}{2} - \frac{1}{2} \|P_i^1 \rho_{\zeta + \tau_i} - P_i^0 \rho_{\zeta}\|, \quad (\text{S11})$$

where $\|\cdots\|$ denotes to the trace norm. The minimum error probability for equally likely hypotheses is

$$\mathcal{P}_e^{el}(\zeta_i, \zeta_i + \tau_i) = \frac{1}{2} - \frac{1}{4} \|\rho_{\zeta + \tau_i} - \rho_{\zeta}\| \geq \frac{1}{2} - \frac{1}{2} \sqrt{1 - F^2(\rho_{\zeta}, \rho_{\zeta + \tau_i})}. \quad (\text{S12})$$

where $F(\rho, \sigma) = \text{Tr}(\sqrt{\rho^{1/2} \sigma \rho^{1/2}})$ and the equal holds for pure states. Therefore, we obtain a lower bound for vector parameter estimation:

$$|\Delta_{\Sigma} \mathbf{x}| \geq \sum_{i=1}^d \int_0^\infty d\tau_i \tau_i \mathcal{V} \int d\zeta \min\{P_{\mathbf{x}}(\zeta), P_{\mathbf{x}}(\zeta + \tau_i)\} \frac{1}{2} \left(1 - \sqrt{1 - F^2(\rho_{\zeta}, \rho_{\zeta + \tau_i})} \right) \quad (\text{S13})$$

which we call a quantum Ziv-Zakai bound for vector parameter estimation.

II. $D = d + 1$ MODE SQUEEZED VACUUM STATE

Using the Taylor expansion

$$\exp(-r\tilde{A}) = c_0(r)\mathbb{I} + c_1(r)\tilde{A} + c_2(r)\tilde{A}^2 + \cdots + c_d(r)\tilde{A}^d, \quad (\text{S14})$$

where $\tilde{A}_{kj}^m = \delta_{(k+m) \bmod D, j}$ and $A_{kj}^m = \delta_{k, (j+m) \bmod D}$, one can obtain that

$$\begin{pmatrix} \exp(-r) \\ \exp(-re^{i\omega_D}) \\ \exp(-re^{i2\omega_D}) \\ \vdots \\ \exp(-re^{id\omega_D}) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{i\omega_D} & e^{i2\omega_D} & \cdots & e^{id\omega_D} \\ 1 & e^{i2\omega_D} & e^{i4\omega_D} & \cdots & e^{i2d\omega_D} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{id\omega_D} & e^{i2d\omega_D} & \cdots & e^{id^2\omega_D} \end{pmatrix} \begin{pmatrix} c_0(r) \\ c_1(r) \\ c_2(r) \\ \vdots \\ c_d(r) \end{pmatrix} \quad (\text{S15})$$

$$\Rightarrow \begin{pmatrix} c_0(r) \\ c_1(r) \\ c_2(r) \\ \vdots \\ c_d(r) \end{pmatrix} = \frac{1}{D} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{-i\omega_D} & e^{-i2\omega_D} & \cdots & e^{-id\omega_D} \\ 1 & e^{-i2\omega_D} & e^{-i4\omega_D} & \cdots & e^{-i2d\omega_D} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{-id\omega_D} & e^{-i2d\omega_D} & \cdots & e^{-id^2\omega_D} \end{pmatrix} \begin{pmatrix} \exp(-r) \\ \exp(-re^{i\omega_D}) \\ \exp(-re^{i2\omega_D}) \\ \vdots \\ \exp(-re^{id\omega_D}) \end{pmatrix} \quad (\text{S16})$$

where $\omega_D = 2\pi/D$. The photon number operator for the k -th mode is $\hat{n}_k = \hat{a}_k^\dagger \hat{a}_k$. The average of the photon number operator is

$$\langle \hat{n}_k \rangle = \langle \mathbf{0} | S^\dagger(r) a_k^\dagger a_k S(r) | \mathbf{0} \rangle = \frac{1}{4} \sum_{i=0}^d [(e^{-r\tilde{A}})_{ki} - (e^{rA})_{ki}]^2 = \frac{1}{4} \sum_{i=0}^d [c_i^2(r) + c_i^2(-r)] - \frac{1}{2} \quad (\text{S17})$$

where $|\mathbf{0}\rangle = |\underbrace{0, \dots, 0}_{d+1}\rangle$, and we have use the fact that $\sum_i (e^{-r\tilde{A}})_{ki} (e^{rA})_{ki} = \sum_i (e^{-r\tilde{A}})_{ki} (e^{r\tilde{A}})_{ik} = 1$. The variance of the photon number operator is calculated as

$$(\Delta \hat{n}_k)^2 = \langle \mathbf{0} | \hat{S}^\dagger(r) \hat{a}_k^\dagger \hat{a}_k \hat{a}_k^\dagger \hat{a}_k \hat{S}(r) | \mathbf{0} \rangle - \langle \mathbf{0} | \hat{S}^\dagger(r) \hat{a}_k^\dagger \hat{a}_k \hat{S}(r) | \mathbf{0} \rangle^2 \quad (\text{S18})$$

$$= \sum_{ijj'j'} \left[\langle \mathbf{0} | T_{ki} R_{kj} R_{kj'} T_{ki'} \hat{a}_i \hat{a}_j \hat{a}_{j'}^\dagger \hat{a}_{i'}^\dagger + T_{ki} T_{kj} T_{kj'} T_{ki'} \hat{a}_i \hat{a}_j^\dagger \hat{a}_{j'}^\dagger \hat{a}_{i'}^\dagger | \mathbf{0} \rangle - T_{ki}^2 T_{kj}^2 \right] \quad (\text{S19})$$

$$= \frac{1}{16} \left(\sum_{i=0}^d [c_i^2(r) + c_i^2(-r)] \right)^2 + \frac{1}{16} \left(\sum_{i=0}^d [c_i^2(r) - c_i^2(-r)] \right)^2 - \frac{1}{4} \quad (\text{S20})$$

$$= \frac{1}{8} \left(\sum_{i=0}^d c_i^2(r) \right)^2 + \frac{1}{8} \left(\sum_{i=0}^d c_i^2(-r) \right)^2 - \frac{1}{4}. \quad (\text{S21})$$

III. QUANTUM METROLOGICAL BOUNDS IN NOISY SYSTEMS

In the noisy evolution cases, we suppose that the impact of the noise in different modes is independent. In the mode $i \in \{1, 2, \dots, d\}$, evolution is determined by the quantum channel expressed in terms of Kraus operators $\hat{\pi}_{l_i}^{(i)}(x_i)$, which satisfies $\sum_{l_i} \hat{\pi}_{l_i}^{(i)\dagger}(x_i) \hat{\pi}_{l_i}^{(i)}(x_i) = \mathbb{I}$. The evolved state is then given by $\rho(\mathbf{x}) = \sum_{\mathbf{l}} \hat{\Pi}_{\mathbf{l}}(\mathbf{x}) \rho \hat{\Pi}_{\mathbf{l}}^\dagger(\mathbf{x})$, where we denote $\mathbf{l} = (l_1, \dots, l_d)$ and $\hat{\Pi}_{\mathbf{l}}(\mathbf{x}) = \hat{\pi}_{l_1}^{(1)}(x_1) \otimes \cdots \otimes \hat{\pi}_{l_d}^{(d)}(x_d)$. Here $\pi_{l_i}^{(i)}(x_i) = {}_E \langle l_i | U_{SE}^{(i)}(x_i) | 0_i \rangle_E$. The fidelity of two states is bounded by Uhlman's theorem

$$\begin{aligned} F^2(\rho_{\mathbf{x}}, \rho_{\mathbf{x}+\boldsymbol{\tau}_i}) &\geq \left| {}_S \langle \psi | {}_E \langle \mathbf{0} | U_{SE}^\dagger(\mathbf{x}) U_{SE}(\mathbf{x} + \boldsymbol{\tau}_i) | \psi \rangle_S | \mathbf{0} \rangle_E \right|^2 \\ &= \left| {}_S \langle \psi | \sum_{\mathbf{l}} \hat{\Pi}_{\mathbf{l}}^\dagger(\mathbf{x}) \hat{\Pi}_{\mathbf{l}}(\mathbf{x} + \boldsymbol{\tau}_i) | \psi \rangle_S \right|^2 = \left| {}_S \langle \psi | \sum_{l_i} \hat{\pi}_{l_i}^\dagger(x_i) \hat{\pi}_{l_i}(x_i + \tau_i) | \psi \rangle_S \right|^2 \end{aligned} \quad (\text{S22})$$

where $|\mathbf{0}\rangle_E = \bigotimes_{i=0}^d |0_i\rangle_E$. $u_E(\mathbf{x})$ connects two purifications of the evolved state. To make the lower bound tighter, we should consider all the possible form of the basis eigenstates of the environment. Then, the effective Hermitian generator for i th mode should be written as[25]

$$\hat{\mathcal{H}}_i(x_i) \equiv -i \frac{d[u_E^{(i)}(x_i) U_{SE}^{(i)}(x_i)]^\dagger}{dx_i} u_E^{(i)}(x_i) U_{SE}^{(i)}(x_i) \quad (\text{S23})$$

with which one obtains the lower bound for the enlarged system when $\hat{\mathcal{H}}_i(x_i)$ is independent of x_i .

$$|\Sigma_\epsilon| \geq \max \left\{ \sum_{i=1}^d \frac{c_{ML}}{\langle \hat{\mathcal{H}}_i \rangle_+}, \sum_{i=1}^d \frac{c_{MT}}{\Delta \hat{\mathcal{H}}_i^2} \right\}. \quad (\text{S24})$$

Tighter bounds are both obtained by figuring out the minimum of effective average $\min \langle \hat{\mathcal{H}}_i \rangle_+$ and variance $\min \Delta \hat{\mathcal{H}}_i^2$ the among all the forms of $u_E(\mathbf{x})$ (forms of Krauss operators [24] or forms of purifications [25], equivalently).

IV. PHOTON LOSS MODEL

For the photon loss model

$$F^2(\rho_{\mathbf{x}}, \rho_{\mathbf{x}+\boldsymbol{\tau}_i}) \geq \left| {}_S \langle \psi | \sum_{l_i} \frac{(1-\eta_i)^{l_i}}{l_i!} (\hat{a}^\dagger)^{l_i} \eta_i^{\hat{n}_i} e^{i\tau_i(\hat{n}_i - \delta_i l_i)} \hat{a}^{l_i} | \psi \rangle_S \right|^2 \geq \left| 1 - \alpha^2 + \alpha^2 \sum_{l_i=0}^N p(l_i) e^{i\tau_i(N - \sigma_i l_i)} \right|^2. \quad (\text{S25})$$

The effective average of the Hermitian operator is $\langle \hat{\mathcal{H}}_i^{pl} \rangle_+ = \langle \hat{n}_i \rangle [1 - \sigma_i(1 - \eta_i)] - \min\{N(1 - \sigma_i), 0\}$ and the variance is $\Delta(\hat{\mathcal{H}}_i^{pl})^2 = \Delta \hat{n}_i^2 [1 - \sigma_i(1 - \eta_i)]^2 + \langle \hat{n}_i \rangle \sigma_i^2 \eta_i (1 - \eta_i)$, where we let $\sigma_i = \delta_i + 1$.

V. PHASE DIFFUSION MODEL

The phase diffusion noise can be represented, in the Markov limit, by the unitary for system and environment under the situation that $\sqrt{2}\beta^2 n \gg 1$ written as[25]

$$u_E(\mathbf{x}) U_{SE}(\mathbf{x}) = \bigotimes_{i=1}^d e^{-ix_i \hat{n}_i} e^{ix_i \kappa_i \hat{P}_i^E / 2\beta_i} e^{i2\beta_i \hat{n}_i \hat{Q}_i^E} \quad (\text{S26})$$

$$\begin{aligned} &= \bigotimes_{i=1}^d \left[\sum_{n_i} |n_i\rangle \langle n_i| e^{-ix_i n_i} \mathcal{D}_E \left(\frac{-x_i \kappa_i}{2\sqrt{2}\beta_i} \right) \mathcal{D}_E(i\sqrt{2}\beta_i n_i) \right] \\ &= \bigotimes_{i=1}^d \left[\sum_{n_i} |n_i\rangle \langle n_i| e^{-ix_i n_i} \mathcal{D}_E \left(i\sqrt{2}\beta_i n_i - \frac{x_i \kappa_i}{2\sqrt{2}\beta_i} \right) \exp \left(\frac{ix_i \kappa_i n_i}{2} \right) \right] \end{aligned} \quad (\text{S27})$$

where $\hat{Q}_E = \frac{\hat{b} + \hat{b}^\dagger}{\sqrt{2}}$, $\hat{P}_E = \frac{\hat{b} - \hat{b}^\dagger}{i\sqrt{2}}$, $\mathcal{D}_E(\alpha)$ is the displacement operator on the environment and we use the fact that $\mathcal{D}_E(\alpha)\mathcal{D}_E(\beta) = \mathcal{D}_E(\alpha + \beta) \exp(i\text{Im}\{\alpha\beta^*\})$. The reduced state is

$$\rho_{\mathbf{x}} = \text{Tr}_E(|\Psi\rangle_{SE} \langle \Psi|) = \sum_{\mathbf{mnl}} \sqrt{P_{\mathbf{m}} P_{\mathbf{n}} E} |\mathbf{l}\rangle u_E(\mathbf{x}) U_{SE}(\mathbf{x}) |\mathbf{0}\rangle_E |\mathbf{m}\rangle \langle \mathbf{n}|_E \langle \mathbf{0}| U_{SE}^\dagger(\mathbf{x}) u_E^\dagger(\mathbf{x}) |\mathbf{l}\rangle_E \quad (\text{S28})$$

where $|\Psi\rangle_{SE} \equiv |\psi\rangle |\mathbf{0}\rangle_E = \sum_{\mathbf{n}} \sqrt{P_{\mathbf{n}}} |\mathbf{n}\rangle |\mathbf{0}\rangle_E = \sum_{n_0 \dots n_d} \sqrt{P_{n_0 \dots n_d}} |n_0 \dots n_d\rangle |\mathbf{0}\rangle_E$.

In this model, the effective Hermitian operator for the enlarged system is $\hat{\mathcal{H}}_i^{pd} = (1 - \kappa_i) \hat{n}_i - \kappa_i \hat{P}_i^E / (2\beta_i)$ and the eigenstate of this Hermitian operator is $\bigotimes_{i=0}^d |n_i\rangle |p_j\rangle_E$ where $p_j \in (-\infty, \infty)$. Therefore, the ground energy E_0 for this Hamiltonian does not exists. In order to obtain the appropriate form of $\langle \hat{\mathcal{H}}_i^{pd} \rangle_+$, we restart from the form of the fidelity:

$$F^2(\rho_{\mathbf{x}}, \rho_{\mathbf{x}+\boldsymbol{\tau}_i}) \geq \left| \sum_{\mathbf{n}} \int_{-\infty}^{\infty} dp_i P_{\mathbf{n}} |\langle \mathbf{n} | \psi \rangle|^2 \frac{1}{\sqrt{\pi}} e^{-p_i^2} e^{-i\tau_i \left[(1-\kappa_i)n_i - \frac{\kappa_i p_i}{2\beta_i} \right]} \right|^2 \quad (\text{S29})$$

$$= \sum_{n_i m_i} \int_{-\infty}^{\infty} dp_i dq_i \frac{|c_{n_i} c_{m_i}|^2}{\pi} e^{-p_i^2 - q_i^2} \cos \left\{ \tau_i \left[(1 - \kappa_i)(n_i - m_i) - \frac{\kappa_i(p_i - q_i)}{2\beta_i} \right] \right\} \quad (\text{S30})$$

$$\geq 1 - \lambda \tau_i \sum_{n_i m_i} \int_{-\infty}^{\infty} dp_i dq_i \frac{|c_{n_i} c_{m_i}|^2}{\pi} e^{-p_i^2 - q_i^2} \left| (1 - \kappa_i)(n_i - m_i) - \frac{\kappa_i(p_i - q_i)}{2\beta_i} \right| \quad (\text{S31})$$

$$\geq 1 - \lambda \tau_i \sum_{n_i m_i} |c_{n_i} c_{m_i}|^2 |(1 - \kappa_i)(n_i - m_i)| + \int_{-\infty}^{\infty} dp_i dq_i \frac{1}{\pi} e^{-p_i^2 - q_i^2} \left| \frac{\kappa_i(p_i - q_i)}{2\beta_i} \right| \quad (\text{S32})$$

$$\geq 1 - 2\lambda \tau_i \langle \hat{\mathcal{H}}_i^{pd} \rangle_+ \quad (\text{S33})$$

where $|c_{n_i}|^2 = \sum_{n_0, \dots, n_{i-1}, n_{i+1}, \dots, n_d} P_{\mathbf{n}} |\langle \mathbf{n} | \psi \rangle|^2$,

$$\langle \hat{\mathcal{H}}_i^{pd} \rangle_+ = |1 - \kappa_i| \langle \hat{n}_i \rangle + \frac{|\kappa_i|}{4\beta_i} \int_{-\infty}^{\infty} \frac{dp_i dq_i}{\pi} e^{-p_i^2 - q_i^2} |p_i - q_i| = |1 - \kappa_i| \langle \hat{n}_i \rangle + \frac{|\kappa_i|}{2\sqrt{2\pi}\beta_i} \quad (\text{S34})$$

Given the values of β_i and $\langle \hat{n}_i \rangle$ the tighter bound is obtained by the optimal κ_i that makes $\langle \hat{\mathcal{H}}_i^{pd} \rangle_+$ minimum.

For the MT bound, it is easy to obtain that [25]

$$\langle \hat{\mathcal{H}}_i^{pd} \rangle = (1 - \kappa_i) \langle \hat{n}_i \rangle \quad (\text{S35})$$

$$(\Delta \hat{\mathcal{H}}_i^{pd})^2 = \Delta \hat{n}_i^2 (1 - \kappa_i)^2 + \frac{\kappa_i^2}{8\beta_i^2} \quad (\text{S36})$$

Considering that all the diffusion parameters are the same $\beta_i = \beta$ for $i = 1, \dots, d$, one obtains the tighter bounds with $\kappa_i = 8\Delta n_i^2 \beta^2 / (1 + 8\Delta n_i^2 \beta^2)$.
